

On monotonicity and order-preservation for multidimensional G -diffusion processes

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Abstract

In this paper, we prove a comparison theorem for multidimensional G -SDEs. Moreover we obtain the equivalent conditions of the monotonicity and order-preservation for two multidimensional G -diffusion processes.

Keywords: G -diffusion processes, G -SDE, Comparison theorem, Monotonicity, Order-preservation.

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1 Introduction

In the classical framework, it is known that Itô diffusions can be used to construct linear semigroups, which are known as Markov semigroups. The relationships among Itô's diffusions, Markov semigroups and infinitesimal generators have been well studied and many interesting results have been deduced. They can be summarized as follows. We suppose $(X_t)_{t \geq 0}$ to be n -dimensional Itô diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion and b, σ are Lipschitz continuous functions on \mathbb{R}^n . The Markov semigroup P_t is defined by $P_t f(x) = E[f(X_t^{0,x})]$, where $X_t^{0,x}$ represents the Itô process with initial condition x at initial time $t = 0$ and f is a function defined on \mathbb{R}^n . Here $E[\cdot]$ stands for the expectation related to a probability P . The infinitesimal generator L of the Markov semigroup, which satisfies

$$Lf(x) = \lim_{t \rightarrow 0^+} \frac{P_t f(x) - f(x)}{t}$$

for f appropriately taken such that the above limit exists, is of the following form:

$$Lf = \frac{1}{2} \text{tr}[\sigma^* \frac{\partial^2}{\partial x^2} f \sigma] + b^* \frac{\partial}{\partial x} f,$$

where " $*$ " denotes the transposition. For more details, the readers can refer to, for example, Stroock and Varadhan ([28]) and Rogers and Williams ([27]).

Herbst and Pitt ([6]) investigated the use of diffusion equations as a tool for establishing stochastic monotonicity of semigroups. Chen and Wang ([2]) continued the study on the order-preservation for multidimensional diffusion processes. One of the main results of Chen and Wang ([2]), which covers the monotonicity result in Herbst and Pitt ([6]), is as follows:

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Lemma 1.1 (Chen and Wang ([2], Theorem 1.3)) Let $A = \frac{1}{2} \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n l_i \frac{\partial}{\partial x_i}$ (resp. $\bar{A} = \frac{1}{2} \sum_{i,j=1}^n \bar{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \bar{l}_i \frac{\partial}{\partial x_i}$). Assume that (a_{ij}) and (\bar{a}_{ij}) are nonnegative definite everywhere, $a_{ij}, \bar{a}_{ij}, l_i, \bar{l}_i \in C(\mathbb{R}^n)$ and the martingale problems for A and \bar{A} are well posed. Let P_t (resp. \bar{P}_t) be the Markov semigroup generated by A (resp. \bar{A}). Then $P_t \geq \bar{P}_t$ if and only if the following two conditions hold:

- (a) for all i and j , $a_{ij} \equiv \bar{a}_{ij}$ and $a_{ij}(x)$ depends only on x_i and x_j ;
- (b) for all i , $l_i(x) \geq \bar{l}_i(y)$ whenever $x_i = y_i$ and $x_j \geq y_j$ for all $j \in \{1, \dots, n\}$ and $j \neq i$.

Peng ([19]) introduced the notion of g -expectation defined via a backward stochastic differential equation (BSDE). A g -expectation preserves most properties of the classical expectations except non-linearity since it is a nonlinear functional. Its nonlinearity can be characterized by its generator g . Zhang and Jia ([29]) constructed a nonlinear semigroup by decoupled FBSDEs. They obtained the equivalent conditions of the monotonicity and order-preservation of semigroups.

Recently, Peng systemically established a time-consistent fully nonlinear expectation theory (see [20], [21] and [22]). As a typical and important case, Peng introduced the G -expectation theory (see [24] and the references therein) in 2006. In the G -expectation framework (G -framework for short), the notion of G -Brownian motion and the corresponding stochastic calculus of Itô's type were established. On that basis, Gao [4] and Peng [23] have studied the existence and uniqueness of the solution of G -SDEs under a standard Lipschitz condition on its coefficients. Moreover, based on Gao [4], Bai and Lin [1] obtained the existence and uniqueness of the solution of GSDEs under some integral-Lipschitz conditions. Lin ([13],[14]) gave some properties of GSDEs and studied the differentiability of GSDEs respectively. Luo and Wang ([16]) studied the sample solutions of G -SDEs by and extension of G -Itô formula. They proved that the integration of a stochastic differential equation driven by G -Brownian motion in \mathbb{R} can be reduced to the integration of an ordinary differential equation parametrized by a variable in (Ω, \mathcal{F}) . Moreover, they obtained a comparison theorem for G -SDEs. Lin ([12]) defined the infinitesimal generator of G -SDEs and obtained the representation theorem under the Lipschitz condition. Recently, proved an existence and uniqueness result on BSDEs driven by G -Brownian motions (G -BSDEs), further (in [9]) they gave a comparison theorem for G -BSDEs. He et. al. ([7]) proved the representation theorem for generators of G -BSDEs, and then the converse comparison theorem of G -BSDEs and some equivalent results for nonlinear expectations generated by G -BSDEs.

This paper is organized as follow: In section 2, we recall some notations and results that we will use in this paper. In section 3, we give our assumptions and recall some notations and results of G -SDEs and G -BSDEs. In section 4, we obtain a comparison theorem for multidimensional G -SDEs. In section 5, we obtain the equivalent conditions of the monotonicity and order-preservation of two multidimensional G -diffusion processes.

2 Preliminaries

2.1 Sublinear expectation, G -Brownian motion and capacity

2.1.1 Sublinear expectation

We present some preliminaries in the theory of sublinear expectation, G -Brownian motions and the capacity under G -framework. More details can be found in Peng [23] and Li and Peng [11].

Definition 2.1 Let Ω be a given set and let \mathcal{H} be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c , and $|X| \in \mathcal{H}$, if $X \in \mathcal{H}$. \mathcal{H} is considered as the space of our random variables. A sublinear expectation \mathbb{E} on \mathcal{H} is a functional $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) Monotonicity: if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

- (b) *Constant preserving*: $\hat{\mathbb{E}}[c] = c, \forall c \in \mathbb{R}$.
(c) *Sub-additivity*: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$.
(d) *Positive homogeneity*: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a *sublinear expectation space*. $X \in \mathcal{H}$ is called a *random variable* in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We often call $Y = (Y_1, \dots, Y_d), Y_i \in \mathcal{H}$ a *d-dimensional random vector* in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. Let us consider a space of random variables \mathcal{H} satisfying: if $X_i \in \mathcal{H}, i = 1, \dots, d$, then $\varphi(X_1, \dots, X_d) \in \mathcal{H}$, for all $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, where $C_{b,Lip}(\mathbb{R}^d)$ is the space of all bounded real-valued Lipschitz continuous functions.

Definition 2.2 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ an *m-dimensional random vector* $X = (X_1, \dots, X_m)$ is said to be *independent from another n-dimensional random vector* $Y = (Y_1, \dots, Y_n)$ under $\hat{\mathbb{E}}$ if for any test function $\varphi \in C_{b,Lip}(\mathbb{R}^{m+n})$ we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$$

Definition 2.3 Let X_1 and X_2 be two *n-dimensional random vector* defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$, respectively. They are called *identically distributed*, denoted by $X_1 \stackrel{d}{=} X_2$, if

$$\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \forall \varphi \in C_{b,Lip}(\mathbb{R}^n)$$

We call \bar{X} an *independent copy* of X if $\bar{X} \stackrel{d}{=} X$ and \bar{X} is independent from X .

Definition 2.4 (G-normal distribution) An *m-dimensional random vector* $X = (X_1, \dots, X_m)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called (*centralized*) *G-normal distributed* if for any $a, b \geq 0$

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2} X$$

where \bar{X} is an independent copy of X . The letter *G* denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[(AX, X)] : \mathbb{S}(d) \mapsto \mathbb{R}.$$

2.1.2 G-Brownian motion

Definition 2.5 (G-Brownian motion) Let $G : \mathbb{S}(d) \mapsto \mathbb{R}$ be a given monotonic and sublinear function. A process $(B(t) \in \mathcal{H})_{t \geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a *G-Brownian motion* if the following properties are satisfied:

- (a) $B(0) = 0$.
(b) For each $t, s \geq 0$ the increment $B_{t+s} - B_t \stackrel{d}{=} \sqrt{s} X$ and independent from $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ for each $n \in \mathbb{N}, 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq t$, where X is *G-normal distributed*.

We denote by $\Omega = C_0^d(\mathbb{R}^+)$ the space of all \mathbb{R}^d -value continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [(\max_{t \in [0, i]} |\omega^1(t) - \omega^2(t)|) \wedge 1]$$

We denote by $\mathcal{B}(\Omega)$ the Borel σ -algebra of Ω .

We also denote, for each $t \in [0, \infty)$:

$$\Omega_t := \{\omega(\cdot \wedge t) : \omega \in \Omega\},$$

$$\mathcal{F}_t := \mathcal{B}(\Omega_t),$$

$L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$ -measurable real function,
 $L^0(\Omega_t)$: the space of all $\mathcal{B}(\Omega_t)$ -measurable real function,
 $B_b(\Omega_t)$: all bounded elements in $L^0(\Omega)$, $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$,
 $C_b(\Omega_t)$: all continuous elements in $B_b(\Omega)$, $C_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t)$.

In Peng [23], a G -Brownian motion is constructed on a sublinear expectation space $(\Omega, \mathbb{L}_G^p, \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t \geq 0})$ for $p = 1$, where \mathbb{L}_G^p is a Banach space under the natural norm $\|X\|_p = \hat{\mathbb{E}}[|X|^p]^{1/p}$. In this space the corresponding canonical process $B(t, \omega) = \omega(t)$, $t \in [0, \infty)$, for $\omega \in \Omega$, is a G -Brownian motion. It is proved in Denis et al.[3] that $L^0(\Omega) \supset \mathbb{L}_G^p(\Omega) \supset C_b(\Omega)$, and there exists a weakly compact family \mathcal{P} of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad X \in \mathbb{L}_G^1(\Omega).$$

We now introduce the natural Choquet capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 2.6 A set $A \subset \Omega$ is polar if $c(A) = 0$. A property holds ‘quasi-surely’ (q.s.) if it holds outside a polar set.

Definition 2.7 A real function X on Ω is said to be quasi-continuous if for each $\varepsilon > 0$, there exists an open set O with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.

Then $\mathbb{L}_G^p(\Omega)$ can be characterized as follows:

$$\mathbb{L}_G^p(\Omega) = \{X \in L^0(\Omega) \mid \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty, \text{ and } X \text{ is c quasi-surely continuous}\}.$$

We denote, for $p > 0$,

$$\mathcal{L}^p := \{X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|^p] = \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\};$$

$$\mathcal{N}^p := \{X \in L^0(\Omega) : \hat{\mathbb{E}}[|X|^p] = 0\};$$

$$\mathcal{N} := \{X \in L^0(\Omega) : X=0, \text{ c-quasi surely (q.s.)}\}$$

It is seen that \mathcal{L}^p and \mathcal{N}^p are linear spaces and $\mathcal{N}^p = \mathcal{N}$, for each $p > 0$. We define the space $L^p(\Omega) = \mathcal{L}^p / \mathcal{N}$ as the equivalence classes of \mathcal{L}^p modulo equality in $\|\cdot\|_p$. Similarly, we can define $L^p(\Omega_t) = L^p(\Omega) \cap L^0(\Omega_t)$. As usual, we do not make the distinction between classes and their representatives.

Definition 2.8 Let $L_b^p(\Omega)$ be the completion of $B_b(\Omega)$ under the Banach norm $(\hat{\mathbb{E}}[|X|^p])^{1/p}$.

Then we have the following characterisation (see [3]): for each $p \geq 1$,

$$L_b^p(\Omega) = \{X \in L^p(\Omega) : \lim_{n \rightarrow \infty} \hat{\mathbb{E}}[|X|^p \mathbf{I}_{\{|X| > n\}}] = 0\}.$$

Definition 2.9 For $p \geq 1$ and $T \in \mathbb{R}^+$ be fixed. Consider the following simple type of processes:

$$M_G^{0,p}(0, T) = \{\eta := \eta(t, \omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1})}(t)$$

$$\forall N > 0, 0 = t_0 < \dots < t_N = T, \xi_j(\omega) \in \mathbb{L}_G^p(\Omega_{t_j}), j = 0, 1, 2, \dots, N-1\}.$$

For each $p \geq 1$, we denote by $M_G^p(0, T)$ the completion of $M_G^{0,p}(0, T)$ under the norm

$$\|\eta\|_{M_G^p(0, T)} = |\hat{\mathbb{E}}[\int_0^T |\eta(t)|^p dt]|^{1/p}.$$

Definition 2.10 For $p \geq 1$ and $T \in \mathbb{R}^+$ be fixed. Consider the following simple type of processes:

$$M_b^0(0, T) = \{\eta := \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{I}_{[t_j, t_{j+1})}(t). \\ \forall N > 0, 0 = t_0 < \dots < t_N = T, \xi_j(\omega) \in B_b(\Omega_{t_j}), j = 0, 1, 2, \dots, N-1.\}.$$

For each $p \geq 1$, we denote by $M_b^p(0, T)$ the completion of $M_b^0(0, T)$ under the norm $\|\cdot\|_{M_G^p(0, T)}$.

Definition 2.11 (Integration with respect to G -Brownian motion) For each $\eta \in M_G^{0,2}(0, T)$ with the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t)$$

define

$$I(\eta) = \int_0^T \eta(s) dB(s) := \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}^N} - B_{t_k^N}).$$

The mapping $I : M_G^{0,2}(0, T) \mapsto \mathbb{L}_G^2(\Omega_T)$ can be continuously extended to $I : M_G^2(0, T) \mapsto \mathbb{L}_G^2(\Omega_T)$. For each $\eta \in M_G^2(0, T)$, the stochastic integral is defined by

$$I(\eta) := \int_0^T \eta(s) dB_s, \quad \eta \in M_G^2(0, T).$$

We have the following general case.

Definition 2.12 For each $\eta \in M_b^0(0, T)$ with the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{I}_{[t_k, t_{k+1})}(t),$$

define

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}^N} - B_{t_k^N}).$$

The mapping $\mathbf{I} : M_b^0(0, T) \mapsto L^2(\Omega_T)$ can be continuously extended to $\mathbf{I} : M_b^2(0, T) \mapsto L^2(\Omega_T)$. For each $\eta \in M_b^2(0, T)$, the stochastic integral is defined by

$$I(\eta) := \int_0^T \eta(s) dB_s, \quad \eta \in M_b^2(0, T).$$

For notational simplicity, we denote by $B^i := B^{e_i}$ the i th coordinate of the d -dimensional G -Brownian motion B , under a given orthonormal basis (e_1, \dots, e_d) of \mathbb{R}^d . We also denote by $B_t^a := (a, B_t)$ for fixed $a \in \mathbb{R}^d$. Then $(B_t^a)_{t \geq 0}$ is a 1-dimensional G -Brownian motion with $\sigma_{aa^T}^2 = \hat{\mathbb{E}}[(a, B_1)^2]$ and $\sigma_{-aa^T}^2 = -\hat{\mathbb{E}}[-(a, B_1)^2]$. Let a and \bar{a} be two given vectors in \mathbb{R}^d . We can define

$$\langle B^a \rangle_t := (B_t^a)^2 - \int_0^t B_s^a dB_s^a$$

where $\langle B^a \rangle$ is called the quadratic variation process of B^a . We can also define mutual variation process by

$$\langle B^a, B^{\bar{a}} \rangle_t := \frac{1}{4} [\langle B^a + B^{\bar{a}} \rangle_t - \langle B^a - B^{\bar{a}} \rangle_t]$$

Itô's integral with respect to $\langle B^a \rangle$ or $\langle B^i, B^j \rangle$ can be similarly defined. By Li and Peng [11], we have

Lemma 2.13 *Let $X \in M_b^p(0, T)$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\eta \in M_b^0(0, T)$ satisfying $\hat{\mathbb{E}} \int_0^T |\eta_t| dt \leq \delta$ and $|\eta_t(\omega)| \leq 1$, we have $\hat{\mathbb{E}} \int_0^T |X_t|^p |\eta_t| dt \leq \varepsilon$.*

Definition 2.14 *A stopping time τ relative to the filtration (\mathcal{F}_t) is a map on Ω with values in $[0, T]$, such that for every t ,*

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Lemma 2.15 *For each stopping time $\tau \in [0, T]$, we have $I_{[0, \tau]}(\cdot)X \in M_b^p(0, T)$, for each $X \in M_b^p(0, T)$.*

Definition 2.16 *A process $(M_t)_{t \geq 0}$ is called a G -martingale if for each $t \in [0, T]$, $M_t \in \mathbb{L}_G^1(\Omega_t)$ and for each $s \in [0, t)$, we have*

$$\hat{\mathbb{E}}_s[M_t] = M_s.$$

Corollary 2.17 *For each $\eta \in M_G^2(0, T)$, the process $(\int_0^t \eta(s) dB_s)_{t \in [0, T]}$ is a G -martingale.*

3 G -SDEs and G -BSDEs

3.1 G -SDEs

We make use of the following assumptions on the generator b, h and σ of GSDE:

- (H1) b, h_{ij} and σ_i are given \mathbb{R}^n -valued bounded continuous functions defined on $[0, T] \times \mathbb{R}^n$ which satisfy the Lipchitz condition, i.e., there exists some constant K such that $|\varphi(t, x) - \varphi(t, y)| \leq K|x - y|$, for each $t \in [0, T]$, $x, y \in \mathbb{R}^n$, $\varphi = b, h_{ij}$ and σ_i respectively, $i, j = 1, \dots, d$.
- (H2) b, h_{ij} and σ_i are given \mathbb{R}^n -valued bounded Lipchitz continuous functions defined on \mathbb{R}^n , i.e., there exists some constant K such that $|\varphi(x) - \varphi(y)| \leq K|x - y|$, for each $x, y \in \mathbb{R}^n$, $\varphi = b, h_{ij}$ and σ_i respectively, $i, j = 1, \dots, d$.

We consider the following SDE driven by a d -dimensional G -Brownian motion:

$$X(t) = X(0) + \int_0^t b(s, X(s)) ds + \int_0^t h_{ij}(s, X(s)) d\langle B^i, B^j \rangle_s + \int_0^t \sigma_i(s, X(s)) dB_s^i, \quad t \in [0, T], \quad (1)$$

where the initial condition $X_0 \in \mathbb{R}^n$ is a given constant.

Theorem 3.1 *Under the assumption (H1), there exists a unique solution $X \in M_G^2(0, T)$ of the stochastic differential equation (1).*

3.2 G -BSDEs

In this subsection, we present some notations and results of G -BSDEs. More details can be found in [8] and [9].

Definition 3.2 *For fixed $T > 0$, let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \dots, t_N\} = \pi_T$ of $[0, T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t),$$

where $\xi_j \in L_{ip}(\Omega_{t_j})$, $j = 0, 1, 2, \dots, N-1$. For $p \geq 1$, we denote by $H_G^p(0, T)$, $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norms $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$ respectively.

For each $\eta \in M_G^1(0, T)$, we can define the integrals $\int_0^T \eta_t dt$ and $\int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}} \rangle_t$ for each $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$. For each $\eta \in H_G^p(0, T; \mathbb{R}^d)$ with $p \geq 1$, we can define Itô's integral $\int_0^T \eta_t dB_t$.

Let $S_G^0(0, T) = \{h(t, B_{t_1 \wedge t}, \dots, B_{t_n \wedge t}) : t_1, \dots, t_n \in [0, T], h \in C_{b, Lip}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0, T)$, set $\|\eta\|_{S_G^p} = \{\mathbb{E}[\sup_{t \in [0, T]} |\eta_t|^p]\}^{\frac{1}{p}}$. Denote by $S_G^p(0, T)$ the completion of $S_G^0(0, T)$ under the norm $\|\cdot\|_{S_G^p}$.

We only consider non-degenerate G -normal distribution, i.e.,

(H3) There exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \text{tr}[A - B]$ for any $A \geq B$.

We consider the following type of G -BSDEs (in this paper we always use Einstein convention):

$$\begin{aligned} Y_t = & \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s \\ & - \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned} \quad (2)$$

where

$$f(t, \omega, y, z), g_{ij}(t, \omega, y, z) : [0, T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

satisfy the following properties:

(H4) There exists some $\beta > 1$ such that for any y, z , $f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$.

(H5) There exists some $L > 0$ such that

$$|f(t, y, z) - f(t, y', z')| + \sum_{i,j=1}^d |g_{ij}(t, y, z) - g_{ij}(t, y', z')| \leq L(|y - y'| + |z - z'|).$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0, T)$ the collection of processes (Y, Z, K) such that $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$, K is a decreasing G -martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 3.3 Let $\xi \in L_G^\beta(\Omega_T)$ and f satisfy (H4) and (H5) for some $\beta > 1$. A triplet of processes (Y, Z, K) is called a solution of equation (2) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y, Z, K) \in \mathfrak{S}_G^\alpha(0, T)$;

(b) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g_{ij}(s, Y_s, Z_s) d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$.

Theorem 3.4 ([8]) Assume that $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H4) and (H5) for some $\beta > 1$. Then equation (2) has a unique solution (Y, Z, K) . Moreover, for any $1 < \alpha < \beta$ we have $Y \in S_G^\alpha(0, T)$, $Z \in H_G^\alpha(0, T; \mathbb{R}^d)$ and $K_T \in L_G^\alpha(\Omega_T)$.

We consider the following G -BSDEs:

$$\begin{aligned} Y_t^{l,\xi} = & \xi + \int_t^T f^l(s, Y_s^{l,\xi}, Z_s^{l,\xi}) ds + \int_t^T g_{ij}^l(s, Y_s^{l,\xi}, Z_s^{l,\xi}) d\langle B^i, B^j \rangle_s \\ & - \int_t^T Z_s^{l,\xi} dB_s - (K_T^{l,\xi} - K_t^{l,\xi}), \quad l = 1, 2, \end{aligned}$$

where $g_{ij}^l = g_{ji}^l$.

He and Hu [7] generalized the comparison theorem in [9].

Proposition 3.5 Let f^l and g_{ij}^l satisfy (H4) and (H5) for some $\beta > 1$, $l = 1, 2$. If $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$, then for each $\xi \in L_G^\beta(\Omega_T)$, we have $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for $t \in [0, T]$.

3.3 Nonlinear Feynman-Kac Formula

In this section, we give the nonlinear Feynman-Kac Formula which was studied in Peng [24] for special type of G -BSDEs. We consider the following type of G -FBSDEs:

$$dX_s^{t,\xi} = b(s, X_s^{t,\xi})ds + h_{ij}(s, X_s^{t,\xi})d\langle B^i, B^j \rangle_s + \sigma_j(s, X_s^{t,\xi})dB_s^j, \quad X_t^{t,\xi} = \xi, \quad (3)$$

$$\begin{aligned} Y_s^{t,\xi} &= \Phi(X_T^{t,\xi}) + \int_s^T f(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})dr + \int_s^T g_{ij}(r, X_r^{t,\xi}, Y_r^{t,\xi}, Z_r^{t,\xi})d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^T Z_r^{t,\xi}dB_r - (K_T^{t,\xi} - K_s^{t,\xi}), \end{aligned} \quad (4)$$

where $b, h_{ij}, \sigma_j : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$, $f, g_{ij} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ are deterministic functions and satisfy the following conditions:

- (A1) $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$ for $1 \leq i, j \leq d$;
- (A2) $b, h_{ij}, \sigma_j, f, g_{ij}$ are continuous in t ;
- (A3) There exist a positive integer m and a constant $L > 0$ such that

$$\begin{aligned} |b(t, x) - b(t, x')| + \sum_{i,j=1}^d |h_{ij}(t, x) - h_{ij}(t, x')| + \sum_{j=1}^d |\sigma_j(t, x) - \sigma_j(t, x')| &\leq L|x - x'|, \\ |\Phi(x) - \Phi(x')| &\leq L(1 + |x|^m + |x'|^m)|x - x'|, \\ |f(t, x, y, z) - f(t, x', y', z')| + \sum_{i,j=1}^d |g_{ij}(t, x, y, z) - g_{ij}(t, x', y', z')| \\ &\leq L[(1 + |x|^m + |x'|^m)|x - x'| + |y - y'| + |z - z'|]. \end{aligned}$$

We define

$$u(t, x) := Y_t^{t,x}, \quad (t, x) \in [0, T] \times \mathbb{R}^n.$$

Remark 3.6 It is important to note that $u(t, x)$ is a deterministic function of (t, x) , because $b, h_{ij}, \sigma_j, \Phi, f, g_{ij}$ are deterministic functions and $\tilde{B}_s := B_{t+s} - B_t$ is a G -Brownian motion.

We now give the Feynman-Kac formula.

Theorem 3.7 Let $u(t, x) := Y_t^{t,x}$ for $(t, x) \in [0, T] \times \mathbb{R}^n$. Then $u(t, x)$ is the unique viscosity solution of the following PDE:

$$\begin{cases} \partial_t u + F(D_x^2 u, D_x u, u, x, t) = 0, \\ u(T, x) = \Phi(x), \end{cases} \quad (5)$$

where

$$\begin{aligned} F(D_x^2 u, D_x u, u, x, t) &= G(H(D_x^2 u, D_x u, u, x, t)) + \langle b(t, x), D_x u \rangle \\ &\quad + f(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle), \\ H_{ij}(D_x^2 u, D_x u, u, x, t) &= \langle D_x^2 u \sigma_i(t, x), \sigma_j(t, x) \rangle + 2\langle D_x u, h_{ij}(t, x) \rangle \\ &\quad + 2g_{ij}(t, x, u, \langle \sigma_1(t, x), D_x u \rangle, \dots, \langle \sigma_d(t, x), D_x u \rangle). \end{aligned}$$

4 Comparison Theorem for Multidimensional GSDEs

In the classical framework, Geiß and Manthey ([5]) obtained a comparison theorem for multidimensional SDEs. We follow the idea of their proof to get our results. We consider the following SDEs driven by a d -dimensional G -Brownian motion:

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t h_{ij}(s, X(s))d\langle B^i, B^j \rangle_s + \int_0^t \sigma_i(s, X(s))dB_s^i, \quad t \in [0, T],$$

and

$$Y(t) = Y(0) + \int_0^t \bar{b}(s, Y(s))ds + \int_0^t \bar{h}_{ij}(s, Y(s))d\langle B^i, B^j \rangle_s + \int_0^t \sigma_i(s, Y(s))dB_s^i, \quad t \in [0, T],$$

where the initial conditions $X(0), Y(0) \in \mathbb{R}^n$ are given constants together with

$$X_k(0) \leq Y_k(0), \quad k = 1, \dots, n$$

We now give a comparison theorem for multidimensional G -SDEs.

Theorem 4.1 *Suppose that the following two conditions hold.*

(B1) *For any $t \in [0, T]$, and $i = 1, \dots, n$ the inequality*

$$b_i(t, x) - \bar{b}_i(t, y) + G([(h_{lk})_i + (h_{kl})_i]_{l,k=1}^d(t, x) - [(\bar{h}_{lk})_i + (\bar{h}_{kl})_i]_{l,k=1}^d(t, y)) \leq 0$$

are fulfilled, whenever $x_i = y_i$ and $x_j \leq y_j$ for all $j \neq i$.

(B2) *If b, h_{ij}, σ_i and $\bar{b}, \bar{h}_{ij}, \sigma_i$ satisfy (H1) and $(\sigma_i)_k$ depends only on x_k , for each $k = 1, \dots, n$, $i, j = 1, \dots, d$ i.e.,*

$$|(\sigma_i)_k(t, x) - (\sigma_i)_k(t, y)| \leq K|x_k - y_k|$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}^n$.

Then for all $t \in [0, T]$,

$$X_k(t) \leq Y_k(t) \quad k = 1, \dots, n \quad q.s.$$

Proof. We first proof the theorem under the following condition (B1') instead of (B1).

(B1') *For any $t \in [0, T]$ and $i = 1, \dots, n$ the inequality*

$$b_i(t, x) - \bar{b}_i(t, y) + G([(h_{lk})_i + (h_{kl})_i]_{l,k=1}^d(t, x) - [(\bar{h}_{lk})_i + (\bar{h}_{kl})_i]_{l,k=1}^d(t, y)) < 0$$

are fulfilled, whenever $x_i = y_i$ and $x_j \leq y_j$ for all $j \neq i$.

Define the stopping times

$$\tau_j = \inf\{t > 0 : X_j(t) > Y_j(t)\} \wedge T, \quad j = 1, \dots, n$$

and

$$\tau := \tau_1 \wedge \dots \wedge \tau_n$$

Obviously, $X_j(\tau_j) = Y_j(\tau_j)$ and $X_j(\tau) \leq Y_j(\tau)$, $j = 1, \dots, n$. Because of condition (B1'), the continuity of $b, h_{ij}, \bar{b}, \bar{h}_{ij}$, $i, j = 1, 2, \dots, d$ and the continuity of X and Y there exists a stopping time $T \geq \kappa > \tau$ q.s. defined on $\{\tau < T\}$ such that

$$\begin{aligned} & b_j(t, X(s)) - \bar{b}_j(t, Y_1(s), \dots, Y_{j-1}(s), X_j(s), Y_{j+1}(s), \dots, Y_d(s)) + G([(h_{lk})_j + (h_{kl})_j]_{l,k=1}^d(t, X(s)) \\ & - [(\bar{h}_{lk})_j + (\bar{h}_{kl})_j]_{l,k=1}^d(t, Y_1(s), \dots, Y_{j-1}(s), X_j(s), Y_{j+1}(s), \dots, Y_d(s))) < 0 \end{aligned} \tag{6}$$

on $\{\tau_j = \tau < T\}$ for all $s \in [\tau, \kappa]$ q.s.. Actually, we can define

$$\begin{aligned} \kappa_1 := \inf \{t > \tau : & b_j(t, X(s)) - \bar{b}_j(t, Y_1(s), \dots, Y_{j-1}(s), X_j(s), Y_{j+1}(s), \dots, Y_d(s)) \\ & + G([(h_{lk})_j + (h_{kl})_j]_{l,k=1}^d(t, X(s)) \\ & - [(\bar{h}_{lk})_j + (\bar{h}_{kl})_j]_{l,k=1}^d(t, Y_1(s), \dots, Y_{j-1}(s), X_j(s), Y_{j+1}(s), \dots, Y_d(s))) > 0\} \wedge T \end{aligned}$$

then we take $\kappa = \frac{\tau + \kappa_1}{2}$. It is easily to check that κ satisfies the above condition.

Now we define $\rho(x) = Kx$ for $x \geq 0$. Then $\int_{0+} \rho^{-2}(u)du = \infty$. Hence there exists a strictly increasing sequence $\{a_n\}_{n=0}^\infty$ such that $a_0 = 1$, $\lim_{n \rightarrow \infty} a_n = 0$ (actually, $a_n = \frac{2}{n(n+1)K^2+2}$) and

$$\int_{a_n}^{a_{n-1}} \rho^{-2}(u)du = n, \quad \text{for all } n \geq 1.$$

Let ψ_n be a continuous function such that its support is contained in (a_n, a_{n-1}) , $0 \leq \psi_n(u) \leq 2\rho^{-2}(u)n^{-1}$ and $\int_{a_n}^{a_{n-1}} \psi_n(u)du = 1$. Put

$$\varphi_n(x) = \begin{cases} 0, & x \leq 0, \\ \int_0^x \int_0^y \psi_n(u)dudy, & x > 0 \end{cases}$$

One can easily see that φ_n is twice continuously differentiable, $\varphi_n(0) = 0$ for $x \leq 0$, $0 \leq \varphi'_n(x) \leq 1$ and $\varphi_n(x) \uparrow x^+$ as $n \rightarrow \infty$. Assume

$$c(\{\tau < T\}) > 0 \tag{7}$$

An application of G -Itô's formula (see [11]) yields

$$\begin{aligned} & \varphi_n(X_k((\tau+t) \wedge \kappa) - Y_k((\tau+t) \wedge \kappa)) \\ = & \varphi_n(X_k(\tau) - Y_k(\tau)) \\ & + \int_\tau^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s)) [b_k(s, X(s)) - \bar{b}_k(s, Y(s))] ds \\ & + \int_\tau^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s)) [(h_{ij})_k(s, X(s)) - (\bar{h}_{ij})_k(s, Y(s))] d\langle B^i, B^j \rangle_s \\ & + \int_\tau^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s)) [(\sigma_i)_k(s, X(s)) - (\sigma_i)_k(s, Y(s))] dB_s^i \\ & + \frac{1}{2} \int_\tau^{(\tau+t) \wedge \kappa} \varphi''_n(X_k(s) - Y_k(s)) [(\sigma_i)_k(s, X(s)) - (\sigma_i)_k(s, Y(s))]^2 d\langle B^i \rangle_s \\ = & \varphi_n(X_k(\tau) - Y_k(\tau \wedge T)) \\ & + \int_\tau^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s)) [b_k(s, X(s)) - \bar{b}_k(s, Y(s))] ds \\ & + G([(h_{ij})_k + (h_{ji})_k]_{i,j=1}^d(s, X(s)) - [(\bar{h}_{ij})_k + (\bar{h}_{ji})_k]_{i,j=1}^d(s, Y(s))] ds \\ & + V_{(\tau+t) \wedge \kappa} - V_\tau \\ & + \int_\tau^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s)) [(\sigma_i)_k(s, X(s)) - (\sigma_i)_k(s, Y(s))] dB_s^i \\ & + \frac{1}{2} \int_{\tau \wedge T}^{(\tau+t) \wedge \kappa} \varphi''_n(X_k(s) - Y_k(s)) [(\sigma_i)_k(s, X(s)) - (\sigma_i)_k(s, Y(s))]^2 d\langle B^i \rangle_s \end{aligned}$$

where

$$V_t = \int_0^t \varphi'_n(X_k(s) - Y_k(s))[(h_{ij})_k(s, X(s)) - (\bar{h}_{ij})_k(s, Y(s))]d\langle B^i, B^j \rangle_s \\ - \int_0^t \varphi'_n(X_k(s) - Y_k(s))G([(h_{ij})_k + (h_{ji})_k]_{i,j=1}^d(s, X(s)) - [(\bar{h}_{ij})_k + (\bar{h}_{ji})_k]_{i,j=1}^d(s, Y(s)))ds$$

It is easy to check that $(V_t)_{0 \leq t \leq T}$ is a decreasing process. Thus

$$\begin{aligned} & \varphi_n(X_k((\tau + t) \wedge \kappa) - Y_k((\tau + t) \wedge \kappa)) \\ & \leq \varphi_n(X_k(\tau) - Y_k(\tau)) \\ & \quad + \int_{\tau}^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s))[b_k(s, X(s)) - \bar{b}_k(s, Y(s)) \\ & \quad + G([(h_{ij})_k + (h_{ji})_k]_{i,j=1}^d(s, X(s)) - [(\bar{h}_{ij})_k + (\bar{h}_{ji})_k]_{i,j=1}^d(s, Y(s)))]ds \\ & \quad + \int_{\tau}^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s))[(\sigma_i)_k(s, X(s)) - (\sigma_i)_k(s, Y(s))]dB_s^i \\ & \quad + \frac{1}{2} \int_{\tau}^{(\tau+t) \wedge \kappa} \varphi''_n(X_k(s) - Y_k(s))[(\sigma_i)_k(s, X(s)) - (\sigma_i)_k(s, Y(s))]^2 d\langle B^i \rangle_s \\ & = S_1(n) + S_2(n) + S_3(n) + S_4(n) \end{aligned}$$

Obviously, from the construction it follows that $S_1(n) = 0$, $n = 1, 2, \dots$

$$\hat{\mathbb{E}}[S_3(n)\mathbf{1}_{\{\tau_k=\tau\}}] = 0.$$

From (B2) we derive

$$\hat{\mathbb{E}}[|S_4(n)|] \leq \frac{Ct}{n}$$

Relation (6) imply

$$\begin{aligned} & S_2(n)\mathbf{1}_{\{\tau_k=\tau\}} \\ & \leq \mathbf{1}_{\{\tau_k=\tau\}} \int_{\tau}^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s))[b_i(t, X(s)) - \bar{b}_i(t, Y_1(s), \dots, Y_{k-1}(s), X_k(s), Y_{k+1}(s), \dots, Y_d(s)) \\ & \quad + G([(h_{ij})_k + (h_{ji})_k]_{i,j=1}^d(t, X(s)) \\ & \quad - [(\bar{h}_{ij})_k + (\bar{h}_{ji})_k]_{i,j=1}^d(t, Y_1(s), \dots, Y_{k-1}(s), X_k(s), Y_{k+1}(s), \dots, Y_d(s)))]ds \\ & \quad + \mathbf{1}_{\{\tau_k=\tau\}} \int_{\tau}^{(\tau+t) \wedge \kappa} \varphi'_n(X_k(s) - Y_k(s))[\bar{b}_i(t, Y_1(s), \dots, Y_{k-1}(s), X_k(s), Y_{k+1}(s), \dots, Y_d(s)) - \bar{b}_k(s, Y(s)) \\ & \quad + G([(h_{ij})_k + (h_{ji})_k]_{i,j=1}^d(s, Y_1(s), \dots, Y_{k-1}(s), X_k(s), Y_{k+1}(s), \dots, Y_d(s)) - [(\bar{h}_{ij})_k + (\bar{h}_{ji})_k]_{i,j=1}^d(s, Y(s)))]ds \\ & \leq \mathbf{1}_{\{\tau_k=\tau\}} K \int_{\tau}^{(\tau+t) \wedge \kappa} [X_k(s) - Y_k(s)]^+ ds \\ & \leq \mathbf{1}_{\{\tau_k=\tau\}} K \int_0^t [X_k((\tau + s) \wedge \kappa) - Y_k((\tau + s) \wedge \kappa)]^+ ds \end{aligned}$$

Consequently, as $n \rightarrow \infty$ we arrive at

$$\hat{\mathbb{E}}[(X_k((\tau + t) \wedge \kappa) - Y_k((\tau + t) \wedge \kappa))^+ \mathbf{1}_{\{\tau_k=\tau\}}] \leq K \int_0^t \hat{\mathbb{E}}[(X_k((\tau + s) \wedge \kappa) - Y_k((\tau + s) \wedge \kappa))^+ \mathbf{1}_{\{\tau_k=\tau\}}] ds$$

In view of Gronwall's inequality this implies

$$\hat{\mathbb{E}}[(X_k((\tau + t) \wedge \kappa) - Y_k((\tau + t) \wedge \kappa))^+ \mathbf{1}_{\{\tau_k = \tau\}}] = 0$$

and hence

$$X_k((\tau + t) \wedge \kappa) \leq Y_k((\tau + t) \wedge \kappa) \quad q.s.$$

on $\{\tau_k = \tau\}$ for all $t \in [\tau, \kappa]$. This contradicts (7).

Now we consider the condition (C1). Let $\epsilon > 0$ be arbitrarily chosen and define

$$b_k^\epsilon := b_k - \epsilon, \quad k = 1, \dots, n.$$

From (B1) it follows immediately that b^ϵ satisfies condition (B1'). Consequently, we get for the corresponding solutions X^ϵ and Y the relation

$$X_k^\epsilon(t) \leq Y_k(t) \quad q.s.$$

for all $t \in [0, T]$, $k = 1, \dots, n$. Choose a strictly decreasing sequence $(\epsilon_m)_{m \geq 1}$ with $\lim_{m \rightarrow \infty} \epsilon_m = 0$. By the same arguments as above we get

$$X_k^{\epsilon_1}(t) \leq X_k^{\epsilon_2}(t) \leq \dots \leq Y_k(t) \quad q.s.$$

as well as

$$X_k^{\epsilon_1}(t) \leq X_k^{\epsilon_2}(t) \leq \dots \leq X_k(t) \quad q.s.$$

for all $t \in [0, T]$, $k = 1, \dots, n$.

Define

$$\tilde{X}_k(t) := \lim_{m \rightarrow \infty} X_k^{\epsilon_m}(t)$$

for each $t \in [0, T]$. Obviously,

$$\tilde{X}_k(t) \leq Y_k(t) \quad q.s.$$

for all $t \in [0, T]$, $k = 1, \dots, n$. Moreover, we have,

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |X^{\epsilon_m}(t) - X(t)|^2] &= \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t (b^{\epsilon_m}(s, X^{\epsilon_m}(s)) - b(s, X(s)))ds \\ &\quad + \int_0^t (h_{ij}(s, X^{\epsilon_m}(s)) - h_{ij}(s, X(s)))d\langle B^i, B^j \rangle \\ &\quad + \int_0^t (\sigma_i(s, X^{\epsilon_m}(s)) - \sigma_i(s, X(s)))dB_s^i|^2] \\ &\leq 3(\hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t (b^{\epsilon_m}(s, X^{\epsilon_m}(s)) - b(s, X(s)))ds|^2] \\ &\quad + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t (h_{ij}(s, X^{\epsilon_m}(s)) - h_{ij}(s, X(s)))d\langle B^i, B^j \rangle|^2] \\ &\quad + \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |\int_0^t (\sigma_i(s, X^{\epsilon_m}(s)) - \sigma_i(s, X(s)))dB_s^i|^2]) \end{aligned}$$

Applying BDG-inequality, we get

$$\begin{aligned} \hat{\mathbb{E}}[\sup_{0 \leq t \leq T} |X^{\epsilon_m}(t) - X(t)|^2] &\leq C(\epsilon_m^2 + \int_0^T \hat{\mathbb{E}}[|X^{\epsilon_m}(t) - X(t)|^2]dt) \\ &\leq C(\epsilon_m^2 + \int_0^T \hat{\mathbb{E}}[\sup_{0 \leq s \leq t} |X^{\epsilon_m}(s) - X(s)|^2]dt) \end{aligned}$$

By Gronwall's inequality, we have

$$\hat{\mathbb{E}}\left[\sup_{0 \leq t \leq T} |X^{\epsilon_m}(t) - X(t)|^2\right] \leq C|\epsilon_m|^2$$

Up to a subsequence, still denoted as $(X^{\epsilon_m})_{m \geq 1}$, we have

$$\lim_{m \rightarrow \infty} X^{\epsilon_m}(t) = X(t) \quad q.s.$$

for all $t \in [0, T]$. This ends the proof. \blacksquare

5 Stochastic monotonicity and order-preservation

Lin ([12]) defined the infinitesimal generator of G -SDEs and obtained the representation theorem under the Lipschitz condition. Similarly, we can obtain the relationships among G -Itô's diffusions, Markov semigroups and infinitesimal generators. They can be summarized as follows. We suppose $(X_t)_{t \geq 0}$ to be n -dimensional G -Itô diffusion

$$dX_t = X_0 + b(X_s)ds + h_{ij}(X_s)d\langle B^i, B^j \rangle_s + \sigma_j(X_s)dB_s^j, \quad t \in [0, T],$$

where $(B_t)_{t \geq 0}$ is a d -dimensional G -Brownian motion and b, h, σ are Lipschitz continuous functions on \mathbb{R}^n . The Markov semigroup \mathcal{E}_t is defined by $\mathcal{E}_t f(x) = \hat{\mathbb{E}}[f(X_t^{0,x})]$, where $X_t^{0,x}$ represents the G -Itô process with initial condition x at initial time $t = 0$ and f is a function defined on \mathbb{R}^n . The infinitesimal generator L of the Markov semigroup, which satisfies

$$Lf(x) = \lim_{t \rightarrow 0^+} \frac{\mathcal{E}_t f(x) - f(x)}{t}$$

for f appropriately taken such that the above limit exists, is of the following form:

$$Lf = \langle \partial_x f, b \rangle + G(\langle \partial_x f, h \rangle + \langle \partial_{xx}^2 f \sigma, \sigma \rangle)$$

where $\langle \partial_x f, h \rangle + \langle \partial_{xx}^2 f \sigma, \sigma \rangle$ is a $d \times d$ symmetric matrix in $\mathbb{S}^d(\mathbb{R})$, defined by:

$$\begin{aligned} \langle \partial_x f, h \rangle + \langle \partial_{xx}^2 f \sigma, \sigma \rangle &:= [\langle \partial_x f, h_{ij} + h_{ji} \rangle + \langle \partial_{xx}^2 f \sigma_i, \sigma_j \rangle]_{i,j=1}^d \\ L &= \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + G\left(\left[\sum_{i=1}^n (h_{ik} + h_{ki})_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n \sigma_{il} \sigma_{jk} \frac{\partial^2}{\partial x_i \partial x_j}\right]_{l,k=1}^d\right). \end{aligned}$$

Now we introduce the following definitions, which are similar to that in Herbst and Pitt ([6]) and Chen and Wang ([2]). Let " \leq " denote the usual semi-order in \mathbb{R}^n .

(1) A measurable function f is called monotone if

$$f(x) \leq f(\bar{x}) \quad \text{for all } x \leq \bar{x}$$

Denote by \mathcal{M} the set of all bounded Lipschitz continuous monotone functions.

(2) For two semigroups $\{\mathcal{E}_t\}_{0 \leq t \leq T}$ and $\{\bar{\mathcal{E}}_t\}_{0 \leq t \leq T}$, we write $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$, if for all $f \in \mathcal{M}$, for all $x \geq \bar{x}$ and $0 \leq t \leq T$,

$$\mathcal{E}_t f(x) \geq \bar{\mathcal{E}}_t f(\bar{x}).$$

If in addition, $\mathcal{E}_t = \bar{\mathcal{E}}_t$, we call \mathcal{E}_t monotone.

Let

$$Lf = \langle \partial_x f, b \rangle + G(\langle \partial_x f, h \rangle + \langle \partial_{xx}^2 f \bar{\sigma}, \bar{\sigma} \rangle)$$

$$\bar{L}f = \langle \partial_x f, \bar{b} \rangle + G(\langle \partial_x f, \bar{h} \rangle + \langle \partial_{xx}^2 f \bar{\sigma}, \bar{\sigma} \rangle)$$

$$\bar{L}'f = \langle \partial_x f, \bar{b} \rangle + G(\langle \partial_x f, \bar{h} \rangle + \langle \partial_{xx}^2 f \sigma, \sigma \rangle)$$

and let $\{\mathcal{E}_t\}_{0 \leq t \leq T}$, $\{\bar{\mathcal{E}}_t\}_{0 \leq t \leq T}$ and $\{\bar{\mathcal{E}}'_t\}_{0 \leq t \leq T}$ be the semigroup generated by L , \bar{L} and \bar{L}' respectively. And we always assume that b , h_{ij} , σ_i and \bar{b} , \bar{h}_{ij} , $\bar{\sigma}_i$ satisfy (H2) for each $i, j = 1, \dots, d$.

We have the following results.

Theorem 5.1 \mathcal{E}_t is monotone if and only if the following conditions hold:

(C1) for all i, j , $\sigma_{li}\sigma_{kj}$ depends only on x_i and x_j , $l, k = 1, \dots, d$

(C2) for all i , $b_i(x) - b_i(y) + G([\langle h_{l,k} \rangle_i(x) + \langle h_{k,l} \rangle_i(x)]_{l,k=1}^d - [\langle h_{l,k} \rangle_i(y) + \langle h_{k,l} \rangle_i(y)]_{l,k=1}^d) \geq 0$ whenever $x \geq y$ with $x_i = y_i$.

Proof. Suppose (C1) and (C2) hold. By setting $\bar{b} = b$ and $\bar{h} = h$ in Theorem (4.1), we have $\forall x \geq \bar{x}$, $X_t^{0,x} \geq X_t^{0,\bar{x}}$ q.s.. Then by the monotonicity of f , the results follows.

Suppose that \mathcal{E}_t is monotone,

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (\mathcal{E}_t f(x) - f(x)) = \langle \partial_x f, b \rangle + G(\langle \partial_x f, h \rangle + \langle \partial_{xx}^2 f \sigma, \sigma \rangle) \quad (8)$$

(a) For given i , we take $x^{(1)} \leq x^{(2)}$ with $x_i^{(1)} = x_i^{(2)}$ and a sequence of functions $f_m \in \mathcal{M} \cap C_b^\infty$ ($m \in \mathbb{N}$) such that $f_m(x) = (x_i - x_i^{(1)} + 1)^{2m+1}$ in a neighborhood of $\{x^{(1)}, x^{(2)}\}$. We have $\lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{E}_t f_m(x^{(1)}) - f_m(x^{(1)})) \leq \lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{E}_t f_m(x^{(2)}) - f_m(x^{(2)}))$, i.e.,

$$\begin{aligned} & \langle \partial_x f(x^{(1)}), b(x^{(1)}) \rangle + G(\langle \partial_x f(x^{(1)}), h(x^{(1)}) \rangle + \langle \partial_{xx}^2 f(x^{(1)}) \sigma(x^{(1)}), \sigma(x^{(1)}) \rangle) \\ & \leq \langle \partial_x f(x^{(2)}), b(x^{(2)}) \rangle + G(\langle \partial_x f(x^{(2)}), h(x^{(2)}) \rangle + \langle \partial_{xx}^2 f(x^{(2)}) \sigma(x^{(2)}), \sigma(x^{(2)}) \rangle) \end{aligned}$$

Then we get

$$\begin{aligned} & G([\frac{(h_{l,k})_i(x^{(2)}) + (h_{k,l})_i(x^{(2)})}{2m} + \sigma_{li}(x^{(2)})\sigma_{ki}(x^{(2)})]_{l,k=1}^d \\ & - [\frac{(h_{l,k})_i(x^{(1)}) + (h_{k,l})_i(x^{(1)})}{2m} + \sigma_{li}(x^{(1)})\sigma_{ki}(x^{(1)})]_{l,k=1}^d) \\ & \geq G([\frac{(h_{l,k})_i(x^{(2)}) + (h_{k,l})_i(x^{(2)})}{2m} + \sigma_{li}(x^{(2)})\sigma_{ki}(x^{(2)})]_{j,k=1}^d) \\ & - G([\frac{(h_{l,k})_i(x^{(1)}) + (h_{k,l})_i(x^{(1)})}{2m} + \sigma_{li}(x^{(1)})\sigma_{ki}(x^{(1)})]_{l,k=1}^d) \\ & \geq \frac{1}{2m} [b_i(x^{(1)}) - b_i(x^{(2)})] \end{aligned}$$

So $[\sigma_{li}(v)\sigma_{ki}(v)]_{l,k=1}^d \geq [\sigma_{li}(u)\sigma_{ki}(u)]_{l,k=1}^d$ as $m \rightarrow \infty$. Replacing f_m with $(x_i - x_i^{(1)} - 1)^{2m+1}$ in a neighborhood of $\{x^{(1)}, x^{(2)}\}$, we obtain the inverse inequality. Therefore, $[\sigma_{li}(v)\sigma_{ki}(v)]_{l,k=1}^d = [\sigma_{li}(u)\sigma_{ki}(u)]_{l,k=1}^d$

- (b) For given i, j $i \neq j$, we take $x^{(1)} \leq x^{(2)}$ with $x_i^{(1)} = x_i^{(2)}$, $x_j^{(1)} = x_j^{(2)}$ and a sequence of functions $f_m \in \mathcal{M} \cap C_b^\infty$ ($m \in \mathbb{N}$) such that $f_m(x) = (x_i + x_j - x_i^{(1)} - x_j^{(1)} + 1)^{2m+1}$ in a neighborhood of $\{x^{(1)}, x^{(2)}\}$. We have $\lim_{t \rightarrow 0} \frac{1}{t}(\mathcal{E}_t f_m(x^{(1)}) - f_m(x^{(1)})) \leq \lim_{t \rightarrow 0} \frac{1}{t}(\mathcal{E}_t f_m(x^{(2)}) - f_m(x^{(2)}))$, i.e.,

$$\begin{aligned} & \langle \partial_x f(x^{(1)}), b(x^{(1)}) \rangle + G(\langle \partial_x f(x^{(1)}), h(x^{(1)}) \rangle + \langle \partial_{xx}^2 f(x^{(1)}) \sigma(x^{(1)}), \sigma(x^{(1)}) \rangle) \\ & \leq \langle \partial_x f(x^{(2)}), b(x^{(2)}) \rangle + G(\langle \partial_x f(x^{(2)}), h(x^{(2)}) \rangle + \langle \partial_{xx}^2 f(x^{(2)}) \sigma(x^{(2)}), \sigma(x^{(2)}) \rangle) \end{aligned}$$

Then we get

$$\begin{aligned} & G\left(\left[\frac{(h_{l,k})_i(x^{(2)}) + (h_{k,l})_i(x^{(2)}) + (h_{l,k})_j(x^{(2)}) + (h_{k,l})_j(x^{(2)})}{2m} + \sigma_{li}(x^{(2)})\sigma_{kj}(x^{(2)})\right]_{l,k=1}^d\right. \\ & \quad \left.- \left[\frac{(h_{l,k})_i(x^{(1)}) + (h_{k,l})_i(x^{(1)}) + (h_{l,k})_j(x^{(1)}) + (h_{k,l})_j(x^{(1)})}{2m} + \sigma_{li}(x^{(1)})\sigma_{kj}(x^{(1)})\right]_{l,k=1}^d\right) \\ & \geq G\left(\left[\frac{(h_{l,k})_i(x^{(2)}) + (h_{k,l})_i(x^{(2)}) + (h_{l,k})_j(x^{(2)}) + (h_{k,l})_j(x^{(2)})}{2m} + \sigma_{li}(x^{(2)})\sigma_{kj}(x^{(2)})\right]_{l,k=1}^d\right) \\ & \quad - G\left(\left[\frac{(h_{l,k})_i(x^{(1)}) + (h_{k,l})_i(x^{(1)}) + (h_{l,k})_j(x^{(1)}) + (h_{k,l})_j(x^{(1)})}{2m} + \sigma_{li}(x^{(1)})\sigma_{kj}(x^{(1)})\right]_{l,k=1}^d\right) \\ & \geq \frac{1}{2m}[b_i(x^{(1)}) + b_j(x^{(1)}) - b_i(x^{(2)}) - b_j(x^{(2)})] \end{aligned}$$

So $[\sigma_{li}(v)\sigma_{kj}(v)]_{l,k=1}^d \geq [\sigma_{li}(u)\sigma_{kj}(u)]_{l,k=1}^d$ as $m \rightarrow \infty$. Replacing f_m with $(x_i + x_j - x_i^{(1)} - x_j^{(1)} + 1)^{2m+1}$ in a neighborhood of $\{x^{(1)}, x^{(2)}\}$, we obtain the inverse inequality. Therefore, $[\sigma_{li}(v)\sigma_{kj}(v)]_{l,k=1}^d = [\sigma_{li}(u)\sigma_{kj}(u)]_{l,k=1}^d$. Thus (C1) holds.

For given i , we take $x^{(1)} \leq x^{(2)}$ with $x_i^{(1)} = x_i^{(2)}$ and $f_m \in \mathcal{M} \cap C_b^\infty$ ($m \in \mathbb{N}$) such that $f(x) = x_i$ in a neighborhood of $\{x^{(1)}, x^{(2)}\}$. By (8), we have

$$b_i(x^{(1)}) + G\left(\left[\frac{(h_{l,k})_i(x^{(1)}) + (h_{k,l})_i(x^{(1)})}{2m}\right]_{l,k=1}^d\right) \leq b_i(x^{(2)}) + G\left(\left[\frac{(h_{l,k})_i(x^{(2)}) + (h_{k,l})_i(x^{(2)})}{2m}\right]_{l,k=1}^d\right)$$

Thus by the subadditivity, (C2) holds.

■

Theorem 5.2 *If $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$ then the following two conditions hold:*

- (D1) *for all i, j , $\sigma_{il}\sigma_{jk} \equiv \bar{\sigma}_{il}\bar{\sigma}_{jk}$ and $\sigma_{il}\sigma_{jk}$ depends only on x_i and x_j , $l, k = 1, \dots, d$*
- (D2) *for all i , $b_i(x) - \bar{b}_i(y) + G\left(\left[\frac{(h_{l,k})_i(x) + (h_{k,l})_i(x)}{2m}\right]_{l,k=1}^d - \left[\frac{(\bar{h}_{l,k})_i(y) + (\bar{h}_{k,l})_i(y)}{2m}\right]_{l,k=1}^d\right) \geq 0$ whenever $x \geq y$ with $x_i = y_i$.*

To prove the above theorem, we need some Lemmas.

Lemma 5.3 *If $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$, then $Lf(x) \geq \bar{L}(y)$ for all $x \geq y$ and $f \in \mathcal{M} \cap C_b^\infty$ with $f(x) = f(y)$*

Proof. Without loss of generality, assume that $f \geq 0$. Choose $m > 0$ such that $\{z : |z| < m\}$ contains x and y and take $h \in C_b^\infty$ such that

$$1 \geq h(z) = \begin{cases} 1, & \text{if } |z| \leq m \\ 0, & \text{if } |x| \geq m+1 \\ > 0, & \text{otherwise.} \end{cases}$$

Set

$$f_1 = hf + a(1 - h), \quad f_2 = hf$$

where a is a constant larger than the upper bound of f . Then $f_1, f_2 \in C_0^\infty$, $f_1 \geq f \geq f_2$ and $f_1 = f = f_2$ on the set $\{z : |z| < m\}$. Since

$$\mathcal{E}_t f(x) \geq \bar{\mathcal{E}}_t f(y), \quad f(x) = f(y),$$

we have

$$\frac{1}{t}[\mathcal{E}_t f_1(x) - f_1(x)] \geq \frac{1}{t}[\bar{\mathcal{E}}_t f_2(x) - f_2(x)]$$

The assertion now follows by letting $t \downarrow 0$. ■

Lemma 5.4 *If $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$, then (D2) holds.*

Proof. For given i , let $u \leq v$ with $u_i = v_i$. Choose $f \in \mathcal{M} \cap C_b^\infty$ such that in a neighborhood of $\{u, v\}$,

$$f(x) = x_i$$

Then by Lemma (5.3), we get

$$b_i(v) - \bar{b}_i(u) + G([(h_{l,k})_i(v) + (h_{k,l})_i(v)]_{l,k=1}^d - [(\bar{h}_{l,k})_i(u) + (\bar{h}_{k,l})_i(u)]_{l,k=1}^d) \geq 0.$$

■

Lemma 5.5 *If $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$, then (D1) holds.*

Proof. The proof consists of two steps.

- (1) For given i , let $u \leq v$ with $u_i = v_i$. Choose $f_m \in \mathcal{M} \cap C_b^\infty$ ($m \in \mathbb{N}$) such that in a neighborhood of $\{u, v\}$,

$$f_m(x) = (x_i - u_i + 1)^{2m+1}$$

By Lemma (5.3), we have

$$\begin{aligned} & G\left(\left[\frac{(h_{l,k})_i(v) + (h_{k,l})_i(v)}{2m} + \sigma_{il}(v)\sigma_{ik}(v)\right]_{l,k=1}^d - \left[\frac{(\bar{h}_{l,k})_i(u) + (\bar{h}_{k,l})_i(u)}{2m} + \bar{\sigma}_{il}(u)\bar{\sigma}_{ik}(u)\right]_{l,k=1}^d\right) \\ & \geq G\left(\left[\frac{(h_{l,k})_i(v) + (h_{k,l})_i(v)}{2m} + \sigma_{il}(v)\sigma_{ik}(v)\right]_{j,k=1}^d\right) - G\left(\left[\frac{(\bar{h}_{l,k})_i(u) + (\bar{h}_{k,l})_i(u)}{2m} + \bar{\sigma}_{il}(u)\bar{\sigma}_{ik}(u)\right]_{l,k=1}^d\right) \\ & \geq \frac{1}{2m}[\bar{b}_i(u) - b_i(v)] \end{aligned}$$

Since m is arbitrary, we deduce that:

$$[\sigma_{il}(v)\sigma_{ik}(v)]_{l,k=1}^d \geq [\bar{\sigma}_{il}(u)\bar{\sigma}_{ik}(u)]_{l,k=1}^d$$

Replacing f_m with $(x_i - u_i - 1)^{2m+1}$ in the neighborhood of $\{u, v\}$, we obtain the inverse inequality. Therefore $\sigma_{il}(v)\sigma_{ik}(v) = \bar{\sigma}_{il}(u)\bar{\sigma}_{ik}(u)$.

- (2) For given $i \neq j$ and $u \leq v$: $u_i = v_i$, $u_j = v_j$, choose $f_m \in \mathcal{M} \cap C_b^\infty$ ($m \in \mathbb{N}$) such that in a neighborhood of $\{u, v\}$,

$$f_m(x) = (x_i + x_j - u_i - u_j + 1)^{2m+1}$$

By (1) and Lemma (5.3), we get

$$\begin{aligned}
& G\left(\left[\frac{(h_{l,k})_i(v) + (h_{k,l})_i(v) + h_{l,k})_j(v) + (h_{k,l})_j(v)}{2m} + \sigma_{il}(v)\sigma_{jk}(v)\right]_{l,k=1}^d\right. \\
& \quad \left.- \left[\frac{(\bar{h}_{l,k})_i(u) + (\bar{h}_{k,l})_i(u) + \bar{h}_{l,k})_j(u) + (\bar{h}_{k,l})_j(u)}{2m} + \bar{\sigma}_{il}(u)\bar{\sigma}_{jk}(u)\right]_{l,k=1}^d\right) \\
& \geq G\left(\left[\frac{(h_{l,k})_i(v) + (h_{k,l})_i(v) + (h_{l,k})_i(v) + (h_{k,l})_j(v)}{2m} + \sigma_{il}(v)\sigma_{jk}(v)\right]_{j,k=1}^d\right) \\
& \quad - G\left(\left[\frac{(\bar{h}_{l,k})_i(u) + (\bar{h}_{k,l})_i(u) + (\bar{h}_{l,k})_i(u) + (\bar{h}_{k,l})_j(u)}{2m} + \bar{\sigma}_{il}(u)\bar{\sigma}_{jk}(u)\right]_{l,k=1}^d\right) \\
& \geq \frac{1}{2m}[\bar{b}_i(u) + \bar{b}_j(u) - b_i(v) - b_j(v)]
\end{aligned}$$

and so

$$[\sigma_{il}(v)\sigma_{jk}(v)]_{l,k=1}^d \geq [\bar{\sigma}_{il}(u)\bar{\sigma}_{jk}(u)]_{l,k=1}^d$$

Similarly, we have the inverse inequality and hence $\sigma_{il}(v)\sigma_{jk}(v) = \bar{\sigma}_{il}(u)\bar{\sigma}_{jk}(u)$

■

proof of theorem 5.2. The proof follows directly from Lemma(5.4) and Lemma (5.5). ■

Theorem 5.6 $\mathcal{E}_t \geq \bar{\mathcal{E}}'_t$ if and only if the following two conditions hold:

(D3) for all i, j , $\sigma_{il}\sigma_{jk}$ depends only on x_i and x_j , $l, k = 1, \dots, d$

(D4) for all i , $b_i(x) - \bar{b}_i(y) + G([\frac{(h_{l,k})_i(x) + (h_{k,l})_i(x)}{2m}]_{l,k=1}^d - [\frac{(\bar{h}_{l,k})_i(y) + (\bar{h}_{k,l})_i(y)}{2m}]_{l,k=1}^d) \geq 0$ whenever $x \geq y$ with $x_i = y_i$.

Proof. Suppose (D3) and (D4) hold. By Theorem (4.1), we have $\forall x \geq \bar{x}$, $X_t^{0,x} \geq \bar{X}_t^{0,\bar{x}}$ q.s.. Then by the monotonicity of f , the results follows. The necessity follows directly from Theorem (5.2). ■

Theorem 5.7 Assume (H3) and assume that $\sigma\sigma^*$ (or resp. $\bar{\sigma}\bar{\sigma}^*$) is uniformly positive definite, i.e., there exists a constant $\beta > 0$, such that for all $y \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $y^*\sigma(x)\sigma^*(x)y \geq \beta|y|^2$. If one of \mathcal{E}_t and $\bar{\mathcal{E}}_t$ is monotone, then $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$ if and only if

(D5) for all i, j , $\sigma_{il}\sigma_{jk} \equiv \bar{\sigma}_{il}\bar{\sigma}_{jk}$ and $\sigma_{il}\sigma_{jk}$ depends only on x_i and x_j , $l, k = 1, \dots, d$

(D6) for all $x, K \in \mathbb{R}^n$, $K \geq 0$, $K^*(b(x) - \bar{b}(x)) + G([K^*((h_{l,k})_i(x) + (h_{k,l})_i(x))]_{l,k=1}^d - [K^*((\bar{h}_{l,k})_i(y) + (\bar{h}_{k,l})_i(y))]_{l,k=1}^d) \geq 0$.

Proof. First, we suppose that $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$. Then (D5) holds directly from Theorem (5.2). For fixed $\bar{x} \in \mathbb{R}^n$, $K \in \mathbb{R}^n$, $K \geq 0$, we take $f \in \mathcal{M} \cap C_b^\infty$ such that in a neighborhood of \bar{x} , $f(x) = \sum_{i=1}^n K_i(x_i - \bar{x}_i)$. Then we have $\lim_{t \rightarrow 0} \frac{1}{t}(\mathcal{E}_t f(\bar{x}) - f(\bar{x})) \geq \lim_{t \rightarrow 0} \frac{1}{t}(\bar{\mathcal{E}}_t f(\bar{x}) - f(\bar{x}))$. Thus (D6) holds.

We now suppose (D5) and (D6) hold. Without loss of generalization, we assume $\bar{\mathcal{E}}$ is monotone. Here we denote $(\sigma\sigma^*)^{-1}\sigma$ by Σ . Let $f \in \mathcal{M}$ and we consider the following G -SDE and G -BSDEs:

$$\begin{aligned}
\bar{X}_t^{0,x} &= x + \int_0^t \sigma_i(\bar{X}_r^{0,x}) dB_r^i, \quad t \in [0, T] \\
Y_s^{0,x,t,f} &= f(\bar{X}_t^{0,x}) + \int_s^t b^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) Z_r^{0,x,t,f} dr + \int_s^t h_{ij}^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) Z_r^{0,x,t,f} d\langle B^i, B^j \rangle_r \\
&\quad - \int_s^t Z_r^{0,x,t,f} dB_r - (K_s - K_t), \quad s \in [0, t]
\end{aligned}$$

and

$$\begin{aligned} \bar{Y}_s^{0,x,t,f} &= f(\bar{X}_t^{0,x}) + \int_s^t \bar{b}^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) \bar{Z}_r^{0,x,t,f} dr + \int_s^t \bar{h}_{ij}^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) \bar{Z}_r^{0,x,t,f} d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^t \bar{Z}_r^{0,x,t,f} dB_r - (\bar{K}_s - \bar{K}_t), \quad s \in [0, t] \end{aligned}$$

We have the following results: $u(t, x) := \mathcal{E}_t f(x) = Y_0^{0,x,t,f}$ is the unique viscosity solution of the following PDE:

$$\begin{cases} \partial_t u - F(\partial_x u, \partial_{xx}^2 u) = 0, \\ u(0, x) = f(x). \end{cases}$$

where

$$F(\partial_x u, \partial_{xx}^2 u) = \langle \partial_x u, b \rangle + G(\langle \partial_x u, h \rangle + \langle \partial_{xx}^2 u \sigma, \sigma \rangle)$$

$\bar{u}(t, x) := \bar{\mathcal{E}}_t = \bar{Y}_0^{0,x,t,f}$ is the unique viscosity solution of the following PDE:

$$\begin{cases} \partial_t \bar{u} - \bar{F}(\partial_x \bar{u}, \partial_{xx}^2 \bar{u}) = 0, \\ \bar{u}(0, x) = f(x). \end{cases}$$

where

$$\bar{F}(\partial_x \bar{u}, \partial_{xx}^2 \bar{u}) = \langle \partial_x \bar{u}, \bar{b} \rangle + G(\langle \partial_x \bar{u}, \bar{h} \rangle + \langle \partial_{xx}^2 \bar{u} \sigma, \sigma \rangle)$$

By Theorem 6.4.3 in Krylov [10] (see also Theorem 4.4 in Appendix C in Peng [23]), there exists a constant $\alpha \in (0, 1)$ such that for each $\kappa > 0$,

$$\|\bar{u}\|_{C^{1+\alpha/2, 2+\alpha}([\kappa, T] \times \mathbb{R}^n)} < \infty.$$

Let $\hat{u}(t, x) = \bar{u}(T - t, x)$ and apply G -Itô's formula to $\hat{u}(s, \bar{X}_s)$ for $s \in [0, T - \kappa]$

$$\begin{aligned} \hat{u}(s, \bar{X}_s^{0,x}) &= \hat{u}(T - \kappa, \bar{X}_{T-\kappa}^{0,x}) + \int_s^{T-\kappa} \bar{b}^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) \sigma^*(\bar{X}_r^{0,x}) \partial_x \hat{u}(r, \bar{X}_r^{0,x}) dr \\ &\quad + \int_s^{T-\kappa} \bar{h}_{ij}^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) \sigma^*(\bar{X}_r^{0,x}) \partial_x \hat{u}(r, \bar{X}_r^{0,x}) d\langle B^i, B^j \rangle_r \\ &\quad - \int_s^{T-\kappa} \sigma^*(\bar{X}_r^{0,x}) \partial_x \hat{u}(r, \bar{X}_r^{0,x}) dB_r - (K'_{T-\kappa} - K'_s) \end{aligned}$$

where

$$\begin{aligned} K_s &= \frac{1}{2} \int_0^s \sigma^*(\bar{X}_r^{0,x}) \partial_{xx}^2 \hat{u}(r, \bar{X}_r^{0,x}) \sigma(\bar{X}_r^{0,x}) d\langle B \rangle_r + \int_s^{T-\kappa} \bar{h}^*(\bar{X}_r^{0,x}) \Sigma(\bar{X}_r^{0,x}) \sigma^*(\bar{X}_r^{0,x}) \partial_x \hat{u}(r, \bar{X}_r^{0,x}) d\langle B \rangle_r \\ &\quad - \int_0^s G(\langle \partial_x \hat{u}(r, \bar{X}_r^{0,x}), \bar{h}^*(\bar{X}_r^{0,x}) \rangle + \langle \partial_{xx}^2 \hat{u}(r, \bar{X}_r^{0,x}) \sigma(\bar{X}_r^{0,x}), \sigma(\bar{X}_r^{0,x}) \rangle) dr \end{aligned}$$

is a non-increasing G -martingale. By the uniqueness of solutions of G -BSDEs, $\bar{Z}_r^{0,x,t,f} = \sigma^*(X_t^{0,x}) \partial_x \hat{u}(t, X_t^{0,x})$ for all $0 \leq r \leq t \leq T - \kappa$. By the assumption that $\bar{\mathcal{E}}_t$ is monotone, we have \bar{u} (then \hat{u}) is nondecreasing in x . Thus $\partial_x \hat{u} \geq 0$. By condition (D6) and the comparison of G -BSDEs, we have $Y_r^{0,x,t,f} \geq \bar{Y}_r^{0,x,t,f}$ for all $0 \leq r \leq t \leq T - \kappa$. Particularly, we have $\mathcal{E}_t f(x) \geq \bar{\mathcal{E}}_t f(x)$. By the monotonicity of $\bar{\mathcal{E}}_t$, for all $x \geq \bar{x}$, $\mathcal{E}_t f(x) \geq \bar{\mathcal{E}}_t f(x) \geq \bar{\mathcal{E}}_t f(\bar{x})$. Thus $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$ for all $t \in [0, T]$. By continuity, we have $\mathcal{E}_t \geq \bar{\mathcal{E}}_t$ for all $t \in [0, T]$. ■

Acknowledgment

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